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# Landau-type extremal problem for the triple $\|f\|_\infty, \|f'\|_p, \|f''\|_\infty$ on a finite interval ☆

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**Abstract**

For any fixed finite interval $I = [a, b]$ and any $p$, $1 \leqslant p \leqslant \infty$, we prove that every extremal function to the problem

$$\|f\|_{L_\infty(I)} \leqslant 1, \quad \|f''\|_{L_\infty(I)} \leqslant 4, \quad \|f'\|_{L_p(I)} \to \sup$$

is a perfect parabolic spline, whose extrema, excluding one at most, take the absolute value 1.


*Keywords:* Landau type extremal problems; Kolmogorov type inequalities

## 1. Introduction

In this note we consider an extremal problem which goes back to a work by Landau [6]. It concerns estimating the norm of $f'$ over a given interval $I$ in terms of the norm of the function itself and its second derivative. The Landau problem was subsequently generalized to intermediate derivatives of functions from Sobolev spaces $W_r^n(I)$ of any given order $n$ and any interval $I$. A typical example of a problem of this type is the following.

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Let $I$ be a given interval (finite or infinite). We shall use the following notations for the $L_p$-norms:

$$||f||_p = ||f||_{L_p(I)} := \left(\int_I |f(x)|^p\,dx\right)^{1/p}, \quad 1 \leqslant p < \infty,$$
$$||f||_\infty = ||f||_{L_\infty(I)} := \sup_{x\in I} \text{vrae}\ |f(x)|.$$

The extremal problem

$$||f||_q \leqslant M_0, \quad ||f^{(n)}||_r \leqslant M_n, \quad ||f^{(k)}||_p \to \sup, \tag{1.1}$$

for any fixed $k$, $1 \leqslant k < n$, is known as a problem of Landau–Kolmogorov type. It was Kolmogorov who first solved a problem of type (1.1) for every integer $n, k$: $1 \leqslant k < n$. He studied the case $p = q = r = \infty$, $I = \mathbb{R}$. When $n = 2$, (1.1) is usually referred to as the Landau problem. By solving (1.1) we mean the complete characterization of the extremal function.

There are a few cases in which (1.1) was solved for every $n$. With the only exception [3], they all deal with $I = \mathbb{R}$, $I = \mathbb{R}^+$, or with classes of periodic functions. It turns out that the problem is much more difficult for a finite interval. The reason is that in the case of a finite interval $I$, the extremal function depends essentially on the length $|I|$ and, thus, $|I|$ plays the role of an additional parameter.

For example, in the classical case when $p = q = r = \infty$ and $I$ is a finite interval, a complete solution of (1.1) was found for $n = 2, 3$, and partially for $n = 4$ (see [8]). Such results are known for every $n$, but with special values of the ratio $\lambda = |I|^n M_n/M_0$ (see [3]). We have to mention also the paper [7] by Pinkus, where he described a narrow family of perfect splines that contains the extremal functions.

Let us note that if $r = \infty$ in (1.1), then any extremal function is a perfect spline, that is, a function $f \in W^n_\infty(I)$ such that $|f^{(n)}(x)| = M_n$ for every $x \in I$, except at *the knots* (the points where $f^{(n)}$ changes its sign). More about these problems and the corresponding results one can find in the survey [9] and the books [4,5].

In the present work we study (1.1) on a finite interval $I$ with $n = 2$, $q = r = \infty$, $1 \leqslant p < \infty$. The same problem for special values of $\lambda$ was considered in [2]. In contrast to [2], where the extremal function is unique up to symmetry, it turns out that in the general case, (1.1) may admit more than one extremal function.

The ratio $\lambda$ is important in (1.1), because it is invariant with respect to the substitution $g(x) = Af(\alpha x)$. Thus, using a linear transformation one can reduce problem (1.1) to an equivalent form in which two of the parameters $M_0, M_n$ and $|I|$ are fixed and the remaining one is free. In our presentation we prefer to fix $M_0$ and $M_n$.

In order to formulate the main result in this study, let us define

$$\Omega^2(I) := \{f \in W^2_\infty(I)\colon\ ||f||_\infty \leqslant 1,\ ||f''||_\infty \leqslant 4\}.$$

where, as usual,

$$W^2_\infty(I) := \{f\colon f'\text{-loc. abs. continuous},\ ||f||_\infty, ||f''||_\infty < \infty\}.$$

Bounds 1 and 4 appearing in the above definitions are chosen according to the Tchebycheff polynomial $T_2(x) = 2x^2 - 1$ and the interval $[-1, 1]$.

Let $\Phi$ be the class of all functions $\phi(x)$ which are positive, increasing and nonconcave on $(0, \infty)$. In addition, let $\Phi_0 := \{\phi \in \Phi : \phi(0) = 0\}$. Then, for every interval $I$ we define

$$J_\phi(g; I) := \int_I \phi(|g(x)|)\, dx.$$

Note that when $\phi(x) = x^p$ the functional $J_\phi(g; I)$ reduces to the $L_p$-norm $||g||^p_{L_p(I)}$.

We say that a parabolic perfect spline with $m$ knots (see (2.1)), is a *zigzag* spline on $[a, b]$ if it has $m + 1$ local extrema on $(a, b)$. We consider the end points of the interval as points of local extrema too. With these notations, we prove the following

**Theorem 1.1.** *Assume that* $I = [a, b]$ *with* $b - a > 2$ *and* $\phi \in \Phi$. *Then every extremal function of the problem*

$$J_\phi(f'; I) \to \sup, \quad \text{over all } f \in \Omega^2(I) \tag{1.2}$$

*is a zigzag perfect parabolic spline on* $[a, b]$. *Moreover, all its extrema, except the eventual one, equal* 1 *in absolute value.*

Let us remark also that the case $|I| \leqslant 2$ was settled recently in [2].

The proof of the theorem is given in Section 3. All auxiliary results concerning mainly properties of the parabolic splines are presented in Section 2. The class of extremal functions is discussed in Section 4 where we give also some further specifications.

## 2. Preliminaries

Recall that any expression of the form

$$s(x) = a + bx + c\left(x^2 + 2\sum_{i=1}^{m} (-1)^i (x - \eta_i)^2_+\right) \tag{2.1}$$

with real $a, b, c$ and $\eta_1 < \cdots < \eta_m$ is called a *perfect spline of second degree* (i.e., a parabolic perfect spline) with knots at $\{\eta_i\}$. We are going to show that the extremal functions to problem (1.2) belong to this special class. As it is easy to see $s'(x)$ is a continuous piecewise linear function with vertices at the knots $\{\eta_i\}$ and equal slopes of alternating sign. Every spline (2.1) has at most $m + 1$ local extrema on $\mathbb{R}$.

More information about perfect splines (of arbitrary degree) can be found in [1].

To simplify the presentation of the proofs we adopt the following terminology and notations. As usual, $f|_{[\alpha,\beta]}$ denotes the restriction of $f(x)$ on the interval $[\alpha, \beta]$. Every maximal subinterval $[\alpha, \beta] \subset [a, b]$ on which $f(x)$ is monotone will be referred to as an *interval of monotonicity* and, in such a case, $f|_{[\alpha,\beta]}$ is called a *branch of monotonicity* of

$f(x)$. For simplicity, we abbreviate these terms to *m-interval* and *m-branch*, respectively.

If $\xi$ is a point of local extremum of $f$ on $[a,b]$ and $|f(\xi)|<1$, then we call $f(\xi)$ an *incomplete* extremum. Similarly, the term incomplete $m$-branch $f|_{[\alpha,\beta]}$ will be used when $|f(\alpha)|<1$ or $|f(\beta)|<1$. The values at the end points will be called *boundary* local extrema to distinguish them from the *interior* ones. Boundary and interior monotone branches are defined similarly.

If $f(x)$ is a perfect zigzag spline of degree 2, and $L$ and $U$ are two consecutive interior local extrema at the points $\alpha$ and $\beta$, respectively, then, clearly, $f|_{[\alpha,\beta]}$ consists of two equal parts of the parabola $\pm cx^2$ joined smoothly at the midpoint (see Fig. 1). That is, $f|_{[\alpha,\beta]}$ coincides with the function

$$\varphi(L,U,[\alpha,\beta];x) := \begin{cases} L+\sigma|c|(x-\alpha)^2 & \text{if } x\in[\alpha,\frac{\alpha+\beta}{2}], \\ U-\sigma|c|(x-\beta)^2 & \text{if } x\in[\frac{\alpha+\beta}{2},\beta], \end{cases}$$

where $\sigma = \operatorname{sign}(U-L)$ and

$$|c| = 2\frac{|U-L|}{(\beta-\alpha)^2}. \tag{2.2}$$

The spline $\varphi(x)$ plays an essential role in our proof as a comparison function. The next lemma gives an extremal property of $\varphi$.

**Lemma 2.1.** *Assume that the function* $g\in W^2_\infty(\mathbb{R})$ *is monotone on* $[\delta,\gamma]$,

$$g(\delta)=L, \quad g(\gamma)=U, \quad g'(\delta)=g'(\gamma)=0 \quad \textit{and} \quad ||g''||_{L_\infty[\delta,\gamma]}\leqslant 2|c|.$$

*Then, with* $\alpha,\beta$ *satisfying* (2.2) *and* $\varphi=\varphi(L,U,[\alpha,\beta];.)$, *we have*

(1) $|\gamma-\delta|\geqslant|\beta-\alpha|$;
(2) $J_\phi(g';[\delta,\gamma])\leqslant J_\phi(\varphi';[\alpha,\beta]),\ \forall\phi\in\Phi_0$.

*Moreover, the equality in* (1) *holds only for* $g(x+\delta-\alpha)\equiv\varphi(x)$ *on* $[\alpha,\beta]$.

The lemma is a part of Proposition 1 from [2]. As an easy consequence we get

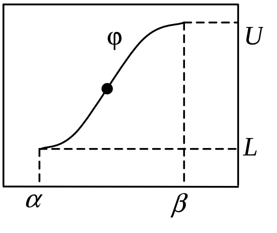


Fig. 1.

**Corollary 2.2.** *Assume that $f(x)$ is a monotone function on $[\delta, \gamma]$ with $f'(\delta) = 0$ and $||f''||_{L_\infty[\delta,\gamma]} \leqslant 4$. Let $\phi \in \Phi_0$ and*

$$P(x) := 2x^2.$$

*Then the equality $|f(\gamma) - f(\delta)| = P(\beta)$ implies*

$$\gamma - \delta \geqslant \beta, \quad \text{and} \quad J_\phi(f'; [\delta, \gamma]) \leqslant J_\phi(P'; [0, \beta]).$$

*Moreover, the equality $\gamma - \delta = \beta$ is attained only if $f(x) = f(\delta) \pm P(x - \delta)$ on $[\delta, \gamma]$.*

The assertion follows immediately from Lemma 2.1 if one continues $P(x)$ and $f(x)$ as odd functions with respect to the points $\beta$ and $\gamma$, respectively.

Because of our normalization in (1.2), in all perfect splines used below $|c| = 2$. We will need the following auxiliary lemma about extension of functions.

**Lemma 2.3.** *Let $f \in \Omega^2([a, b])$ with $b - a \geqslant \sqrt{2}$ and $c = (a + b)/2$. Then there exist functions $g_1 \in \Omega^2((-\infty, b])$ and $g_2 \in \Omega^2([a, \infty))$ such that*

$$f \equiv g_1 \text{ on } [c, b], \quad f \equiv g_2 \text{ on } [a, c].$$

*Moreover, $g_1$, $g_2$ can be chosen to be constants, respectively, on $(-\infty, \delta]$, $[\gamma, \infty)$ and parabolas on $[\delta, c]$, $[c, \gamma]$) with appropriate $\delta$, $\gamma$.*

The lemma is an easy consequence of Lemma 1 from [2] and its proof. There the numbers $\delta$ and $\gamma$ are given explicitly.

**Remark.** It is clear that if $f$ is a monotone function, then $g_1$ and $g_2$ are monotone, too.

## 3. Proof of Theorem 1.1

Let us note first that the existence of an extremal function $f$ of problem (1.2) in $\Omega^2(I)$ follows by standard compactness arguments. Thus, our task below is to characterize $f$.

Note first that in what follows the function $\phi$ is assumed to belong to $\Phi_0$. This is not a restriction since, otherwise we can consider $\phi(x) - \phi(0)$.

The characterization of $f$ goes in three steps, in which we prove the following:

**Claim 1.** *Every extremal function to* (1.2) *is a perfect zigzag spline.*

**Claim 2.** *At least one of two consecutive extrema of $f$ is complete.*

**Claim 3.** *There is no more than one incomplete extremum of $f$.*

The proof of Claim 1 is based on the following simple lemmas.

**Lemma 3.1.** *$f$ is not a constant on any subinterval of $I$.*

**Proof.** Assume that $f$ is a constant on a subinterval $[\alpha, \beta]$ of $I$. Then, a small admissible variation of $f$ by a function, supported on $[\alpha, \beta]$, leads to a contradiction with the extremality of $f$. $\square$

**Lemma 3.2.** *$f$ consists of at least two m-branches on $I$ ($|I|>2$).*

**Proof.** Assume that $f$ is monotone on $I$. Then, we apply the auxiliary results from Section 2 to compare the values of $J_\phi$ for the function $f$ and the spline $s$, defined by (see Fig. 2)

$$s(x) := \begin{cases} -1 + 2(x-a-1)^2 & \text{on } [a, a+1], \\ -1 & \text{on } [a+1, b-1], \\ -1 + 2(x-b+1)^2 & \text{on } [b-1, b]. \end{cases}$$

Let $g_1$ be the extension of $f|_{[c,b]}$, $(c := (a+b)/2)$ on $(-\infty, b]$ in the sense of Lemma 2.3. Then, by Corollary 2.2, we conclude that

$$J_\phi(f'; [c,b]) \leqslant J_\phi({g_1}'; (-\infty, b]) \leqslant J_\phi(s'; [c,b]).$$

Similarly we estimate $J_\phi(f')$ on the left half-interval by $J_\phi(s'; [a,c])$. Therefore, since $s$ is not optimal (it has a constant part) $f$ is also not optimal, a contradiction. $\square$

**Lemma 3.3.** *On each interior m-interval, $f$ coincides with a spline $\varphi$.*

**Proof.** Let $[\delta, \gamma] \subset (a,b)$ be an interior $m$-interval of $f$. We shall show that $f$ must coincide with the parabolic perfect spline $\varphi(x) := \varphi(L, U, [\alpha, \beta]; x)$, where $L = f(\delta)$, $U = f(\gamma)$, $\alpha = \delta$ and $\beta$ is defined according to (2.2) with $|c| = 2$. The functions $\varphi$ and $f|_{[\delta,\gamma]}$ satisfy the conditions of Lemma 2.1. Hence

$$\beta - \alpha \leqslant \gamma - \delta, \quad \text{i.e.,} \quad \beta \leqslant \gamma, \tag{2.3}$$

and

$$J(f'; [\delta, \gamma]) \leqslant J(\varphi'; [\alpha, \beta]).$$

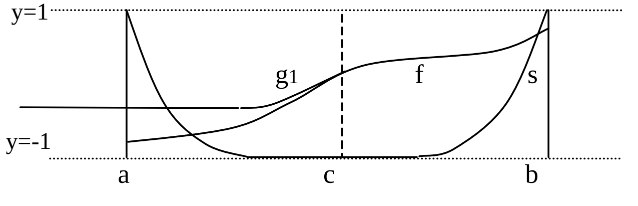


Fig. 2.

If the equality holds in (2.3), then $f(x) \equiv \varphi(x)$ on $[\alpha, \beta] = [\delta, \gamma]$, which was to be shown. If (2.3) is strict, then we consider $\tilde{f}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{on } [a,b] \backslash [\delta, \gamma], \\ \varphi(x) & \text{on } [\alpha, \beta], \\ U = f(\gamma) & \text{on } [\beta, \gamma], \end{cases}$$

and in view of the above inequality, $J(f'; I) \leqslant J(\tilde{f}'; I)$. But $\tilde{f}$ is not an extremal function because it is constant on a subinterval, consequently $f(x)$ is not an extremal function either, a contradiction. □

**Lemma 3.4.** *On a boundary m-interval, f coincides with a translation of the parabola P.*

**Proof.** For definiteness, consider the $m$-branch $f|_{[\delta,b]}$ and assume that it is increasing. As in the previous lemma, we introduce the auxiliary spline $s$,

$$s(x) := \begin{cases} f(\delta) & \text{on } [a, \alpha], \\ f(\delta) + \operatorname{sign}(f(b) - f(\delta)) P(x - \alpha) & \text{on } [\alpha, b], \end{cases}$$

where $\alpha$ is determined such that $s(b) = f(b)$ (see Fig. 3).

Using this time Corollary 2.2, one can easily see that

$$\delta \leqslant \alpha \quad \text{and} \quad J_\phi(f'; [\delta, b]) \leqslant J_\phi(s'; [\alpha, b])$$

which allows us to conclude that $\delta = \alpha$ and $f|_{[\delta,b]} \equiv s|_{[\delta,b]}$. □

**Proposition 3.5** (Claim 1). *The function f is a zigzag spline on I.*

The proof follows immediately from the previous lemmas.

The following improvement of Lemma 3.2 follows on the basis of the observation that $\Omega^2([a,b])$ contains no parabolas if $|I| > 2$.

**Corollary 3.6.** *f consists of at least three m-branches on I.*

For the proof of Claim 2 we need the following lemma.

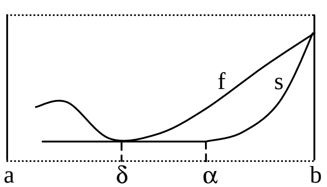


Fig. 3.

**Lemma 3.7.** *Let $s_1^0, s_2^0, s_3^0$ be a solution of the system*

(1) $s_1, s_2, s_3 > 0$;
(2) $s_1 + s_2 + s_3 = C_1$;
(3) $s_1^2 - s_2^2 + s_3^2 = C_2$

*with some fixed $C_1, C_2 \in \mathbb{R}$. Then, in a neighborhood of $s_1^0, s_2^0, s_3^0$ the system has a one-parametric family of solutions.*

**Proof.** Let us denote $s_3 - s_1 = 2t$ and solve the system formally. With $D = t^2 + (C_1^2 - C_2)/2$, we obtain

$$\begin{cases} s_1 = C_1 - t - \sqrt{D}, \\ s_3 = C_1 + t - \sqrt{D}, \\ s_2 = 2\sqrt{D} - C_1. \end{cases} \tag{3.1}$$

The sign of $\sqrt{D}$ can be easily determined from (1) and the last equation of (3.1). Moreover, with $t_0 = (s_3^0 - s_1^0)/2$, we conclude that $D(t_0) > C_1^2/4 > 0$. Consequently, in a neighborhood of $t_0 = (s_3^0 - s_1^0)/2$, equalities (3.1) give a solution of the system, which is unique for a fixed $t$.

**Proposition 3.8** (Claim 2). *Let $[\alpha_2, \beta_2]$ be an m-branch of $f$. Then $|f(\alpha_2)| = 1$ or $|f(\beta_2)| = 1$.*

**Proof.** Let us assume that $|f(\alpha_2)|, |f(\beta_2)| < 1$. In view of our previous considerations $f$ is a zigzag spline with at least three $m$-branches on $[a, b]$. There are two possibilities: the branch $f|_{[\alpha_2,\beta_2]}$ is a boundary or an interior one. We next consider these two cases separately.

*Case* 1: Let $f|_{[\alpha_2,\beta_2]}$ be a boundary $m$-branch. Without loss of generality, we can assume that $\beta_2 = b$ and $f$ is increasing on $[\alpha_2, \beta_2]$. Let $f|_{[\alpha_1,\beta_1]}$ be the preceding[1] monotone branch (that is, $\beta_1 = \alpha_2$). Because of Corollary 3.6, both the branches cannot be boundary. Thus $f'(\alpha_1) = 0$.

Let us set $s_i^0 := \beta_i - \alpha_i,\ i = 1, 2$. Next we introduce new variables $s_1, s_2$ satisfying the relation

$$s_1 + s_2 = b - \alpha_1 =: C.$$

[1] Hypothetically, it can happen that the preceding monotone branch does not exist, for example, when $\alpha_2$ is a density point of the zeros of $f'(x)$. But in this case we can choose $\varepsilon > 0$ such that $f'(\alpha_2 - \varepsilon) = 0$ and rearrange (continuously) the decreasing branches of $f$ on $[\alpha_2 - \varepsilon, \alpha_2]$ in the left part of this interval and the increasing ones—to the right. Then, for sufficiently small $\varepsilon$ we get an admissible function from $\Omega^2(I)$ which has the same value of the functional $J$ as that of $f$ and is not optimal.

Consider $s_1$ in a $\delta$-neighborhood of $s_1^0$ (i.e., $|s_1 - s_1^0| \leqslant \delta$). With $s_1, s_2$ we associate a spline $\tilde{f}(x)$ (which we shall compare with $f$) defined as follows:

$$\tilde{f}(x) := \begin{cases} f(x) & \text{on } [a, \alpha_1], \\ \varphi_1(x) & \text{on } [\alpha_1, \tilde{\beta}_1], \\ \tilde{f}(\tilde{\beta}_1) + P(x - \tilde{\beta}_1) & \text{on } [\tilde{\beta}_1, b], \end{cases}$$

where $\tilde{\beta}_1 := \alpha_1 + s_1$, $\varphi_1(x) := \varphi(f(\alpha_1), f(\alpha_1) - s_1^2, [\alpha_1, \tilde{\beta}_1]; x)$, $\quad P(x) = 2x^2$. Set

$$F(t) := \int_0^t \phi(|P'(x)|)\, dx. \tag{3.2}$$

Then we have

$$\begin{aligned} J(\tilde{f}'; I) &= J(\tilde{f}', [a, \alpha_1]) + J(\tilde{f}', [\alpha_1, \tilde{\beta}_1]) + J(\tilde{f}', [\tilde{\beta}_1, b]) \\ &= J(f', [a, \alpha_1]) + 2F\left(\frac{s_1}{2}\right) + F(s_2). \end{aligned}$$

Now taking into account that $F(t)$ is a convex function and $s_2 = C - s_1$, we obtain

$$\frac{d^2}{ds_1^2} J(\tilde{f}'; I) = \frac{1}{2} F''\left(\frac{s_1}{2}\right) + F''(C - s_1) > 0.$$

This means that the integral $J(\tilde{f}'; I)$ is a convex function with respect to the parameter $s_1$ and, consequently, it cannot attain a local maximum at an interior point $s_1 \in [s_1^0 - \delta, s_1^0 + \delta]$. Hence, $f$ is not extremal, a contradiction. Therefore, the extremal function has no boundary incomplete (from both ends) monotone branch.

*Case* 2: Let $f|_{[\alpha_2, \beta_2]}$ be an interior $m$-branch. Thus, it has a neighboring branch from the left and from the right. Let us denote the whole triple by $\varphi_1, \varphi_2, \varphi_3$ with supports $[\alpha_i, \beta_i]$, $i = 1, 2, 3$, respectively. First, let us consider the case when all of them are interior. Without loss of generality, we may assume that $\varphi_2$ is decreasing and $\varphi_1, \varphi_3$ are increasing functions.

Let us set $s_i^0 := \beta_i - \alpha_i$, $i = 1, 2, 3$. As in the previous case, we introduce the variables $s_1, s_2, s_3$, varying in neighborhoods of $s_1^0, s_2^0, s_3^0$ and satisfying the conditions

$$\begin{cases} s_1 + s_2 + s_3 = \beta_3 - \alpha_1 =: C_1, \\ s_1^2 - s_2^2 + s_3^2 = f(\beta_3) - f(\alpha_1) =: C_2. \end{cases} \tag{3.3}$$

It follows from Lemma 3.7 that system (3.3) has one degree of freedom and the solution can be described uniquely by the parameter $t = (s_3 - s_1)/2$ varying in $[t_0 - \delta, t_0 + \delta]$ with $t_0 = (s_3^0 - s_1^0)/2$ and sufficiently small $\delta$. Any solution gives rise to a perturbation $\tilde{f}$ of $f$ with new monotone branches $\tilde{\varphi}_1$, $\tilde{\varphi}_2$, $\tilde{\varphi}_3$ supported on

$[\tilde{\alpha}_i, \tilde{\beta}_i]$, $i = 1, 2, 3$. The perturbation is defined by

$$\tilde{\alpha}_1 := \alpha_1, \quad \tilde{\alpha}_{i+1} := \tilde{\beta}_i := \tilde{\alpha}_i + s_i, \quad i = 1, 2, 3,$$

$$\tilde{f}(x) := f(x) \quad \text{on } [a, \alpha_1] \cup [\tilde{\beta}_3, b],$$

$$\tilde{\varphi}_1(x) := \varphi(f(\alpha_1), f(\alpha_1) + s_1^2, [\alpha_1, \tilde{\beta}_1]; x), \quad \tilde{f}(x) := \tilde{\varphi}_1(x) \text{ on } [\alpha_1, \tilde{\beta}_1],$$

$$\tilde{\varphi}_2(x) := \varphi(\tilde{f}(\tilde{\alpha}_2), \tilde{f}(\tilde{\alpha}_2) - s_2^2, [\tilde{\alpha}_2, \tilde{\beta}_2]; x), \quad \tilde{f}(x) := \tilde{\varphi}_2(x) \quad \text{on } [\tilde{\alpha}_2, \tilde{\beta}_2],$$

$$\tilde{\varphi}_3(x) := \varphi(\tilde{f}(\tilde{\alpha}_3), \tilde{f}(\tilde{\alpha}_3) + s_3^2, [\tilde{\alpha}_3, \tilde{\beta}_3]; x), \quad \tilde{f}(x) := \tilde{\varphi}_3(x) \quad \text{on } [\tilde{\alpha}_3, \tilde{\beta}_3].$$

In view of (3.3), we have $\tilde{\beta}_3 = \beta_3$ and $\tilde{f}(x) \in W^2_\infty(I)$. Moreover, if $t = t_0$, then $\tilde{f}(x) \equiv f(x)$ on $I$, $\tilde{\alpha}_2 = \alpha_2$, $\tilde{\beta}_2 = \beta_2$ and, consequently,

$$|\tilde{f}(\tilde{\alpha}_2)|, |\tilde{f}(\tilde{\beta}_2)| < 1. \tag{3.4}$$

Because of the continuity, for sufficiently small $\delta$, relation (3.4) holds for every $t \in [t_0 - \delta, t_0 + \delta]$. In other words, the function $\tilde{f}(x)$ is admissible.

Next, we differentiate $R(t) := J(\tilde{f}_t; I)$ and get

$$R'(t) = \frac{d}{dt}\left(\sum_{i=1}^{3} 2F\left(\frac{s_i}{2}\right)\right) = \phi(2s_1)s_1{}' + \phi(2s_2)s_2{}' + \phi(2s_3)s_3{}'. \tag{3.5}$$

From (3.1) we find

$$\begin{cases} s_1{}' = -1 - t/\sqrt{D} = -(s_2 + s_3)/\sqrt{D}, \\ s_3{}' = 1 + t/\sqrt{D} = (s_1 + s_2)/\sqrt{D}, \\ s_2{}' = 2t/\sqrt{D} = (s_3 - s_1)/\sqrt{D}. \end{cases}$$

Substituting these equations into (3.5), we obtain

$$R'(t) = \frac{1}{\sqrt{D}}\{(\phi(2s_3) - \phi(2s_1))s_2 + \phi(2s_2)(s_3 - s_1) + s_1\phi(2s_3) - s_3\phi(2s_1)\}. \tag{3.6}$$

We shall show that $\operatorname{sign}(R'(t)) = \operatorname{sign}(s_3 - s_1)$. Indeed, let $s_3 > s_1$. The summands in (3.6) which involve $s_2$ are obviously positive. In order to estimate the remaining difference, we use the fact that $\phi(x)/x$ is a nondecreasing function for every $\phi \in \Phi_0(\phi(0) = 0)$. Then

$$s_1\phi(2s_3) - s_3\phi(2s_1) = 2s_1s_3\left(\frac{\phi(2s_3)}{2s_3} - \frac{\phi(2s_1)}{2s_1}\right) \geqslant 0.$$

In the case $s_1 > s_3$, the situation is symmetric to the above one, $R'(t) < 0$, while the equality $R'(0) = 0$ is clear.

So, if $2t_0 = s_3^0 - s_1^0 \neq 0$, then $R(t)$ is monotone in a neighborhood of $t_0$, and if $t_0 = 0$, then $R(t)$ has a local minimum. Both possibilities lead to a contradiction with the extremality of $f = f_{t_0}$.

If one or both monotone branches $f|_{[\alpha_1, \beta_1]}$, $f|_{[\alpha_3, \beta_3]}$ are boundary ones, then the reasoning is quite similar to that given above. For instance, if $\beta_3 = b$, then instead of system (3.3) we get

$$\begin{cases} s_1 + s_2 + s_3 = \beta_3 - \alpha_1 =: C_1, \\ s_1^2 - s_2^2 + 2s_3^2 = f(\beta_3) - f(\alpha_1) =: C_2, \end{cases}$$

which, together with $3t = 2s_3 - s_1$ and $D = t^2 + (2C_1^2 - C_2)/3$, leads to

$$\begin{cases} s_1 = 2C_1 - t - 2\sqrt{D}, \\ s_3 = C_1 + t - \sqrt{D}, \\ s_2 = 3\sqrt{D} - 2C_1. \end{cases}$$

Also, the definition of $\tilde{f}(x)$ on $[a, \tilde{\alpha}_3]$ remains the same, while, on $[\tilde{\alpha}_3, b]$, $\tilde{f}(x) = \tilde{f}(\tilde{\alpha}_3) + P(x - \tilde{\alpha}_3)$. Thus, in this case, the following modifications should be made in the calculations:

$$\begin{aligned} R'(t) &= \frac{d}{dt}\left(2F\left(\frac{s_1}{2}\right) + 2F\left(\frac{s_2}{2}\right) + F(s_3)\right) \\ &= \phi(2s_1){s_1}' + \phi(2s_2){s_2}' + \phi(4s_3){s_3}' \\ &= \frac{1}{\sqrt{D}}\{(\phi(4s_3) - \phi(2s_1))s_2 + \phi(2s_2)(2s_3 - s_1) + s_1\phi(4s_3) - 2s_3\phi(2s_1)\} \end{aligned}$$

and we arrive at the same conclusion, namely, $\operatorname{sign}(R'(t)) = \operatorname{sign}(t)$. Consequently $f$ cannot be an extremal function, a contradiction.

Note that the last case just discussed can be reduced to another one, studied already, by an odd reflection of the $m$-branch $f|_{[\alpha_3, b]}$ with respect to $x = b$.

**Proposition 3.9** (Claim 3). *The function $f$ has at most one incomplete m-branch.*

**Proof.** Let $a = t_0 < \cdots < t_n = b$ be the points of local extrema of $f$. According to the results proved already, every $m$-branch of $f$ attains the value 1 or $-1$ at one end-point. Then, we shall distinguish three types of incomplete extrema.

*Type* 0: Those at $t_0$ and $t_n$, see Fig. 5 (d);

*Type* 1: Those at $t_1$ and $t_{n-1}$, see Fig. 5 (e);

*Type* 2: Those at $t_2, \ldots, t_{n-2}$, see Fig. 5 (c).

With any local extremum $f(t_j)$ we associate an interval $[\alpha_j, \beta_j]$, (which we call the *support* of the extremum), as follows: $[\alpha_j, \beta_j] = [t_{j-1}, t_{j+1}]$, for $j = 0, \ldots, n$, where $t_{-1} = a$, $t_{n+1} = b$. Then, the local representation of $f$ on the support $[\alpha, \beta]$ of an extremum of type $i$ can be described by the function $e_i(t; x)$ where $t = \beta - \alpha$:

$$e_0(t; x) = -1 + 2x^2,$$

$$e_1(t; x) = -1 + 2x^2 - 4\left(x - \frac{t}{2 + \sqrt{2}}\right)_+^2,$$

$$e_2(t; x) = -1 + 2x^2 - 4\left(x - \frac{t}{4}\right)_+^2 + 4\left(x - \frac{3t}{4}\right)_+^2.$$

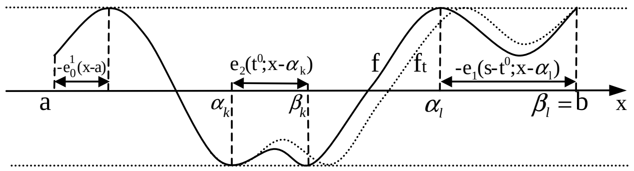


Fig. 4.

Further, let us introduce the notation $e_i^1(t;x) := e_i(t;t-x)$ for the symmetric function of $e_i(t;x) =: e_i^0(t;x)$. With F(t) defined by (3.2), elementary calculations show that

$$J(f';[\alpha,\beta]) = J(e_i{}'(t;.);[0,t]) = \begin{cases} F(t) & \text{for } i=0, \\ 2F(\frac{t}{2+\sqrt{2}}) + F(\frac{\sqrt{2}t}{2+\sqrt{2}}) & \text{for } i=1, \\ 4F(\frac{t}{4}) & \text{for } i=2. \end{cases}$$

In each of these three cases $J$ is a convex function of the parameter $t$.

Let us assume that $f$ has at least two incomplete extrema, say $f(t_k)$ and $f(t_l)$ $(t_k < t_l)$, of type $i, j$, respectively. Now we shall construct a perturbation of $f(x)$ by increasing one incomplete extremum and decreasing the other. Let $[\alpha_k, \beta_k]$ and $[\alpha_l, \beta_l]$ be the supports of these extrema, $t^0 := \beta_k - \alpha_k$ and $s := (\beta_k - \alpha_k) + (\beta_l - \alpha_l)$. Then, with $\sigma := \operatorname{sign}(f''(a))$ and $\tau_0 = \tau_1 = 1$, $\tau_2 = \cdots = \tau_n = 0$ we consider the function

$$f_t(x) \equiv \begin{cases} f(x) & \text{on } [a, \alpha_k) \cup (\beta_l, b], \\ \sigma(-1)^k e_i^{\tau_k}(t; x-\alpha_k) & \text{on } [\alpha_k, \alpha_k + t], \\ f(x - t + t^0) & \text{on } [\alpha_k + t, \alpha_l + t - t^0], \\ \sigma(-1)^l e_j^{\tau_l}(s-t; x - \alpha_l - t + t^0) & \text{on } [\alpha_l + t - t^0, \beta_l]. \end{cases}$$

Here the parameter $t$ changes in $[t^0 - \delta, t^0 + \delta]$ with sufficiently small $\delta$ such that $0 < t < s$ and $|e_i(t;x)|, |e_j(s-t;x)| \leqslant 1$ on $[0,t]$. The function is illustrated in Fig. 4.

Because of the convexity of $R_1(t) := J(e_i{}';[0,t])$ and $R_2(t) := J(e_j{}';[0,s-t])$, the integral $J(f_t{}';[a,b])$ as a function of the parameter $t \in [t^0 - \delta, t^0 + \delta]$ is convex, too. Consequently $f(x) = f_{t^0}(x)$ is not an optimal function. This contradiction completes the proofs of Claim 3 and Theorem 1.1.

## 4. Comments on the extremal functions

At first sight it seems that the theorem describes a wide class of functions as possible solutions to our extremal problem, namely, the functions $\{\pm f_{l,k}\}$, $l = 3, 4, \ldots, k = 0, \ldots, l$, where $l+1$ is the number of local extrema, $k$ is the index of the corresponding incomplete extremum and $f''_{l,k}(a) > 0$. However, if we consider the

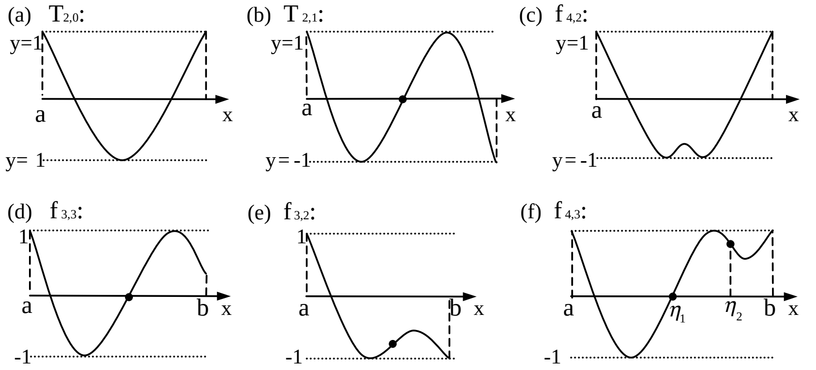


Fig. 5.

graph of $f_{l,k}$ as a composition of its monotone parts (in a given order), we see that many of the candidates are equivalent, that is, they are composed by the same parts. Note also that the number $l$ is determined (up to 1) by the length of the interval:

$$2+\frac{|I|-2}{\sqrt{2}}\leqslant l<4+\frac{|I|-2}{\sqrt{2}}.$$

Thus, the actual number of possible solutions is much smaller, namely 5, which we shall describe explicitly below.

Note first that, without loss of generality, we can assume that the first $m$-branch of the candidate $f_*=f_{l,k}$ is complete, (i.e. $k>1$). Otherwise we can consider $(-1)^l f_{l,k}(a+b-x)=f_{l,l-k}(x)$. Then, if $a=t_0<\cdots<t_l=b$ $(=: t_{l+1})$ are the points of local extremum, and $i$ is the type of $f_*$, we may write explicitly:

if $i=0$, $t_j=a+1+\sqrt{2}(j-1)$, $j=1,\dots,l-1$,

if $i=1$, $t_j=a+1+\sqrt{2}(j-1)$, $j=1,\dots,l-2$, $t_{l-1}=(\sqrt{2}t_{l-2}+t_l)/(\sqrt{2}+1)$,

if $i=2$, $t_j=a+1+\sqrt{2}(j-1)$, $j=1,\dots,k-1$, $t_j=b-1-\sqrt{2}(l-1-j)$, $j=k+1,\dots,l-1$, $\quad t_k=(t_{k-1}+t_{k+1})/2$,

and (see Fig. 5)

$$f_{l,k}(x)=\begin{cases}2(x-a-1)^2-1 & \text{for } x\in[a,a+1],\\ \varphi((-1)^j,(-1)^{j+1},[t_j,t_{j+1}];x) & \text{for } x\in[t_j,t_{j+1}],\ j\in\{1,\dots,l-2\}\backslash\{k-1,k\},\\ (-1)^k e_i(t_{k+1}-t_{k-1};x) & \text{for } x\in[t_{k-1},t_{k+1}],\\ (-1)^l(2(x-t_{l-1})^2-1) & \text{for } x\in[b-1,b] \text{ and } i=2.\end{cases}$$

In view of the comments at the beginning of this section and the explicit expressions above, it is seen that for $|I|\in(2+(m-1)\sqrt{2},2+m\sqrt{2})$, $m\in\mathbb{N}$, the essentially different extremal functions are amongst

$$f_{m+2,m+2},\quad f_{m+2,m+1},\quad f_{m+2,m},\quad f_{m+3,m+2}\quad \text{or}\quad f_{m+3,m+1}.$$

Depending on $|I|$, it could happen that some of the above functions do not exist.

In the particular case (not covered above) when $|I| = 2 + m\sqrt{2}$, we have $f_{m+2,k} = T_{2,m},\ k = 0, \dots, m$, where $T_{2,m}(x)$ is the Tchebycheff perfect spline of second degree, normalized by $T_{2,m}(t_i) = (-1)^i,\ i = 0, \dots, m+2$. As was shown in [2], in this case $T_{2,m}$ is the unique (up to symmetry) extremal function to (1.2).

Consider now in detail the case $J(f) = ||f||^p_{L_p(I)},\ p \geqslant 1$. We have

| Type, $f_*$ | $\lvert I\rvert$ | $4^{-p}(p+1)\lvert\lvert f_*'\rvert\rvert_p^p$ |
|---|---|---|
| 0, $f_{m+2,m+2}$ : | $1 + m\sqrt{2} + z,\ z \in (0, 1]$ | $1 + m2^{(1-p)/2} + z^{p+1}$ |
| 1, $f_{m+3,m+2}$ : | $1 + m\sqrt{2} + z,\ z \in (0, \sqrt{2} + 1]$ | $1 + m2^{(1-p)/2} + z^{p+1}$ |
| 2, $f_{m+4,m+2}$ : | $2 + m\sqrt{2} + z,\ z \in (0, 2\sqrt{2}]$ | $2 + m2^{(1-p)/2} + 4^{-p}z^{p+1}$ |

It is easy to find from this table the extremal $f_{l,k}$ for a given $|I|$.

Careful analysis shows that the following is true.

*Case* A: $|I| \in (2, 2 + \sqrt{2})$. Theorem 1.1 admits as extremal functions $f_{4,2},\ f_{3,3},\ f_{3,2}$ or $f_{4,3}$. It turns out that each of $f_{4,2},\ f_{3,3}$ and $f_{3,2}$ is optimal, depending on $|I|$ and $p$ (see the table below).

*Case* B: $|I| \in (2 + m\sqrt{2}, 2 + (m+1)\sqrt{2}),\ m \in \mathbb{N}$. Then, among the five candidates for an extremal function of (1.1), only $f_{m+4,m+2}$ and $f_{m+3,m+1}$ can play this role.

| Case | Interval | Extremal functions |
|---|---|---|
| | $\lvert I\rvert \in (2, 2\sqrt{2})$ | $f_{4,2}$ |
| A | $\lvert I\rvert \in [2\sqrt{2}, q],\ q = \frac{16+3\sqrt{2}}{7} \approx 2.9$ | $f_{4,2},\ f_{3,3}$ or $f_{3,2}$ |
| | $\lvert I\rvert \in (q, 2 + \sqrt{2})$ | $f_{4,2}$ or $f_{3,2}$ |
| B | $\lvert I\rvert \in (2 + m\sqrt{2},\ 2 + (m + \frac{1}{2})\sqrt{2})$ | $f_{m+4,m+2}$ |
| $(m \in \mathbb{N})$ | $\lvert I\rvert \in (2 + (m + \frac{1}{2})\sqrt{2},\ 2 + (m+1)\sqrt{2})$ | $f_{m+4,m+2}$ or $f_{m+3,m+1}$ |

Note that in the analysis of extremal functions, the theory of rearrangements (see for instance [4]), might be useful. On the basis of this theory, some of the results in the particular case $p = 1$ can be extended to every $p \geqslant 1$. The following assertion reveals a relation between the solutions of (1.1) for different values of the parameter $p$.

**Proposition 4.1.** *For given* $p_1, p_2 < \infty,\ 1 \leqslant p_1 < p_2$, *and a fixed interval* $I$, *let the function* $f_{l,k}$ *be extremal to* (1.2) *with* $p = p_1$ *and* $p = p_2$. *Then* $f_{l,k}$ *is an extremal function for every* $p \in [p_1, p_2]$.

Our proof, although elementary, is quite clumsy. That is why we omit it.

Note also that Proposition 4.1 is no longer true in the case $n = 3$.

Finally, we pose two questions:

(1) Does the analog of Theorem 1.1 hold if $0 < p < 1$?
(2) Can problem (1.2) be solved in a constructive way?

An answer to the second question perhaps will give a way to extend the result of the present paper to the case $n = 3$.

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